Primeri:

1. Slučajna promenljiva X ima gustinu:

$$f(x)=\begin{cases} -x+\sqrt{2}, & x\in\left[0,\sqrt{2}\right] \\ 0, & izvan \end{cases}.$$

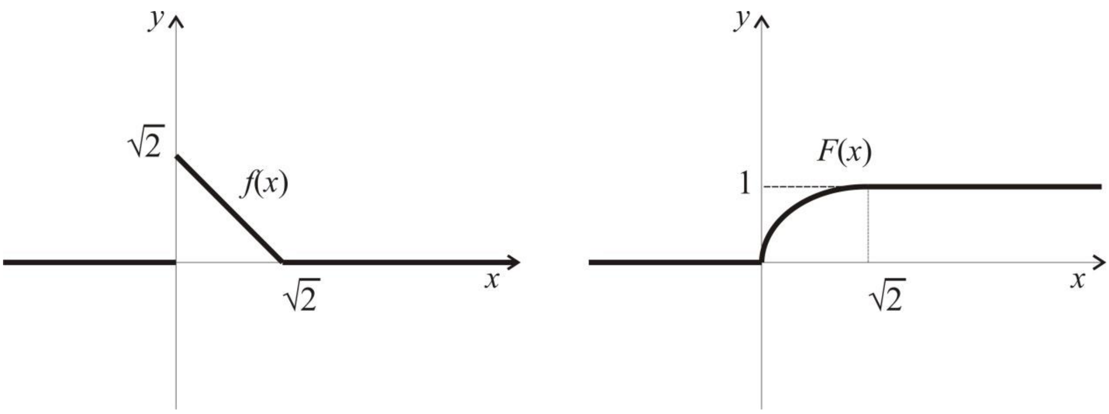
   a) Odrediti funkciju raspodele $F(x)$ i skicirati grafike funkcija $f(x)$ i $F(x)$,

   b) Izračunati $E(X)$, $D(X)$ i $P\left(\frac{\sqrt{2}}{2}<X<\sqrt{2}\right)$.

Rešenje:

a) Kako je $F(x)=\int_{-\infty}^{x} f(x)dx$, imamo:

$$F(x)=\begin{cases} 0, \; x\le 0 \\ \int_0^x\left(-t+\sqrt{2}\right)dt=\left(\frac{-t^2}{2}+\sqrt{2}t\right)\Big|_0^x=-\frac{x^2}{2}+\sqrt{2}x, \; 0<x\le\sqrt{2}. \\ 1, \; x>\sqrt{2} \end{cases}$$



Slika 1.

b) $E(X)=\int_{-\infty}^{+\infty} x\cdot f(x)dx=\int_0^{\sqrt{2}} x\left(-x+\sqrt{2}\right)dx=\int_0^{\sqrt{2}}\left(-x^2+\sqrt{2}x\right)dx=$

$$=\left(-\frac{x^3}{3}+\sqrt{2}\frac{x^2}{3}\right)\Big|_0^{\sqrt{2}}=-1+\frac{4}{3}=\frac{1}{3},$$

$$E\left(X^2\right)=\int_{-\infty}^{+\infty} x^2\cdot f(x)dx=\int_0^{\sqrt{2}} x^2\left(-x+\sqrt{2}\right)dx=\int_0^{\sqrt{2}}\left(-x^3+\sqrt{2}x^2\right)dx=$$

$$=\left(-\frac{x^4}{4}+\sqrt{2}\frac{x^3}{3}\right)\Big|_0^{\sqrt{2}}=-1+\frac{4}{3}=\frac{1}{3},$$

$$D(X)=E\left(X^2\right)-\left(E(X)\right)^2=\frac{1}{3}-\frac{2}{9}=\frac{1}{9},$$

$$P\left(\frac{\sqrt{2}}{2}<X<\sqrt{2}\right)=F\left(\sqrt{2}\right)-F\left(\frac{\sqrt{2}}{2}\right)=1-\frac{3}{4}=\frac{1}{4}.$$

2. Slučajna promenljiva X ima gustinu $f(x)=\begin{cases} k(x+1), & x\in[-1,0] \\ 0, & izvan \end{cases}$.

a) Odrediti konstantu k, funkciju raspodele F i skicirati grafike za f i F.

b) Izračunati $E(X)$, $D(X)$ i $P\left(X>-\frac{1}{2}\right)$.

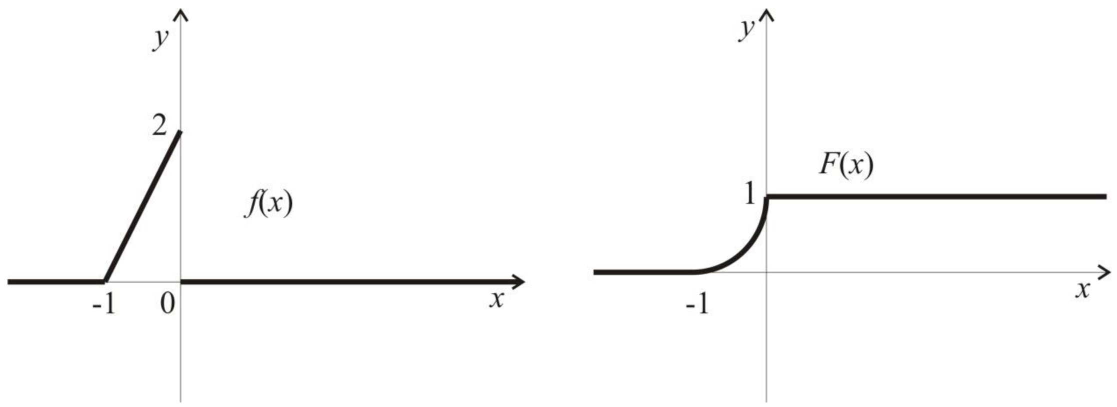
Rešenje:

a) Iz $\int_{-\infty}^{+\infty} f(x)dx=1$, sledi da je $\int_{-1}^{0} k(x+1)dx=k\left(\frac{x^2}{2}+x\right)\Big|_{-1}^{0}=1$, tj. $k=2$, pa je

$$f(x)=\begin{cases} 2x+2, & x\in[-1,0] \\ 0 & izvan \end{cases}.$$

Kako je $F(x)=\int_{-\infty}^{x} f(x)dx$ imamo:

$$F(x)=\begin{cases} 0, \; x\le -1 \\ x^2+2x+1, -1<x\le 0. \\ 1, \; x>1 \end{cases}$$



Slika 2.

b) $E(X)=\int_{-\infty}^{+\infty} xf(x)dx=\int_{-1}^{0} x(2x+2)dx=\int_{-1}^{0}(2x^2+2x)dx=\left(2\cdot\frac{x^3}{3}+2\cdot\frac{x^2}{2}\right)\Big|_{-1}^{0}=-\frac{1}{3}$,

$E(X^2)=\int_{-\infty}^{+\infty} x^2 f(x)dx=\int_{-1}^{0} x^2(2x+2)dx=\int_{-1}^{0}(2x^3+2x^2)dx=\left(2\cdot\frac{x^4}{4}+2\cdot\frac{x^3}{3}\right)\Big|_{-1}^{0}=\frac{1}{6}$,

$D(X)=E(X^2)-(E(X))^2=\frac{1}{6}-\frac{1}{9}=\frac{1}{18}$,

$P\left(X>-\frac{1}{2}\right)=P\left(-\frac{1}{2}<X<+\infty\right)=1-F\left(-\frac{1}{2}\right)=1-\frac{1}{4}=\frac{3}{4}$.

3. Slučajna promenljiva X ima gustinu $f(x)=\begin{cases} a\cos x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2}. \\ 0, & van \end{cases}$

a) Odrediti konstantu a i funkciju raspodele $F(x)$.

b) Izračunati $E(X)$, $D(X)$ i $P\left(\frac{\pi}{4}\le X\le\frac{\pi}{2}\right)$.

Rešenje:

a) Iz $\int_{-\infty}^{+\infty} a\cos x dx=1$, sledi da je $\int_{\frac{-\pi}{2}}^{\pi/2} a\cos x dx = a\sin x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = 2a=1$ tj. $a=\frac{1}{2}$, pa je

$$f(x)=\begin{cases} \frac{1}{2}\cos x, & -\frac{\pi}{2}\le x\le\frac{\pi}{2}. \\ 0, & van \end{cases}$$

Kako je $F(x)=\int_{-\infty}^{x} f(x)dx$, imamo:

$$F(x)=\begin{cases} 0, \; x\le -\frac{\pi}{2} \\ \int_{\frac{-\pi}{2}}^{x} \frac{1}{2}\cos t dt = \frac{1}{2}+\frac{1}{2}\sin x, \; -\frac{\pi}{2}< x\le\frac{\pi}{2}. \\ 1, \; x>\frac{\pi}{2} \end{cases}$$

b) $E(X)=\int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{2}\int_{\frac{-\pi}{2}}^{\pi/2} x\cos x dx = \frac{1}{2}x\sin x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{2}\int_{\frac{-\pi}{2}}^{\pi/2}\sin x dx =$

$$=\frac{1}{2}\cos x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = 0.$$

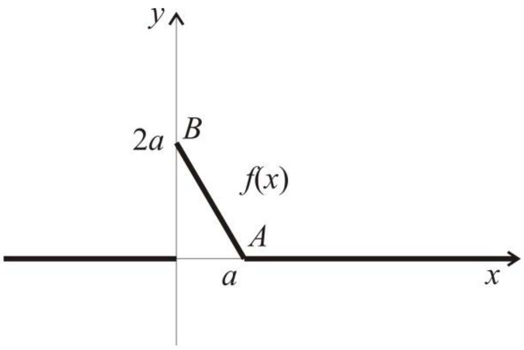
$$E(X^2)=\int_{-\infty}^{+\infty} x^2 f(x)dx = \frac{1}{2}\int_{\frac{-\pi}{2}}^{\pi/2} x^2\cos x dx = \frac{1}{2}x^2\sin x\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} - \int_{\frac{-\pi}{2}}^{\pi/2}\sin x dx =$$

$$=\frac{\pi^2}{8}+\frac{\pi^2}{8}+(x\cos x-\sin x)\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^2}{4}-2 .$$

$$D(X)=\frac{\pi^2}{4}-2$$

$$P\left(\frac{\pi}{4}\le X\le\frac{\pi}{2}\right)=F\left(\frac{\pi}{2}\right)-F\left(\frac{\pi}{4}\right)=\frac{1}{2}+\frac{1}{2}\sin\frac{\pi}{2}-\left(\frac{1}{2}+\frac{1}{2}\sin\frac{\pi}{4}\right)=\frac{1}{2}-\frac{\sqrt{2}}{4}=\frac{2-\sqrt{2}}{4} .$$

4. Gustina slučajne promenljive X data je grafikom:



Slika 3.

a) Odrediti konstantu a, tačan izraz za gustinu, funkciju raspodele F.

b) Izračunati $E(X)$, $D(X)$ i $P\left(X > \frac{a}{3}\right)$.

Rešenje:

a) Površina trougla čija su temena tačke $O(0,0)$, $A(a,0)$ i $B(0,2\text{a})$ je $a^2 = 1$, odakle je $a = 1$. Dakle $f(x) = \begin{cases} -2x+2, & x \in [0,1] \\ 0, & van \end{cases}$

$$F(x) = \begin{cases} 0, x \le 0 \\ \int_0^x (-2t+2)dt = -x^2 + 2x, 0 < x \le -1. \\ 1, x > 1 \end{cases}$$

b) $E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_0^1 x(-2x+2)dx = \int_0^1 (-2x^2+2x)dx =$

$$= \left(-2\frac{x^3}{3} + x^2\right)\Bigg|_0^1 = -\frac{2}{3} + 1 = \frac{1}{3},$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_0^1 x^2(-2x+2)dx = \int_0^1 (-2x^3+2x^2)dx =$$

$$= \left(-2\frac{x^4}{4} + 2\frac{x^3}{3}\right)\Bigg|_0^1 = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6},$$

$$D(X) = E(X^2) - (E(X))^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18},$$

$$P\left(X > \frac{1}{3}\right) = P\left(\frac{1}{3} < X < +\infty\right) = F(+\infty) - F\left(\frac{1}{3}\right) = 1 - \left(-\frac{1}{9} + \frac{2}{3}\right) = \frac{4}{9}.$$

5. Gustina slučajne promenljive X je:

$$f(x)=\begin{cases} \frac{k}{x}, & x\in[1,2] \\ 0, & van \end{cases}.$$

a) Odrediti konstantu k i funkciju raspodele $F(x)$.

b) Izračunati $E(X)$, $D(X)$ i $P(X>\sqrt{2})$.

Rešenje:

a) Iz $\int_{-\infty}^{+\infty}\frac{k}{x}dx=1$, sledi da je $k\cdot\ln|x|\Big|_1^2=1$, tj. $k(\ln 2-\ln 1)=1$, pa je $k=\frac{1}{\ln 2}$, tj.

$$f(x)=\begin{cases} \frac{1}{\ln 2}\cdot\frac{1}{x}, & x\in[1,2] \\ 0, & van \end{cases}.$$

Kako je $F(x)=\int_{-\infty}^{x}f(x)dx$, imamo:

$$F(x)=\begin{cases} 0, x\leq 1 \\ \int_0^x\frac{1}{\ln 2}\cdot\frac{1}{t}dt=\frac{\ln x}{\ln 2}, 1<x\leq 2. \\ 1, x>2 \end{cases}$$

b) $E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\frac{1}{\ln 2}\int_1^2 x\cdot\frac{1}{x}dx=\frac{1}{\ln 2}$,

$$E(X^2)=\int_{-\infty}^{+\infty}x^2f(x)dx=\frac{1}{\ln 2}\int_1^2 x^2\cdot\frac{1}{x}dx=\frac{1}{\ln 2}\cdot\frac{x^2}{2}\Big|_1^2=\frac{1}{\ln 2}\left(2-\frac{1}{2}\right)=\frac{3}{2}\cdot\frac{1}{\ln 2},$$

$$D(X)=E(X^2)-(E(X))^2=\frac{3}{2}\cdot\frac{1}{\ln 2}-\frac{1}{\ln^2 2}=\frac{1}{\ln 2}\cdot\frac{3\ln 2-2}{2\ln 2},$$

$$P(X>\sqrt{2})=P(\sqrt{2}<X<+\infty)=1-F(\sqrt{2})=1-\frac{\ln\sqrt{2}}{\ln 2}=\frac{1}{2}.$$

6. Funkcija raspodele slučajne promenljive X je:

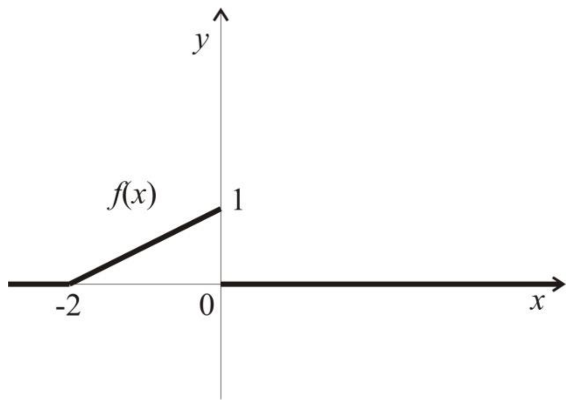
$$F(x)=\begin{cases}0, \ x\le -2\\ \frac{1}{4}(x+2)^2, \ -2<x\le 0.\\ 1, \ x>0\end{cases}$$

a) Odrediti gustinu $f(x)$ za X, skicirati grafik funkcije $f(x)$.

b) Izračunati $E(X)$ i $P(X>-1)$.

Rešenje:

a) Kako je $f(x)=F'(x)$, onda je $f(x)=\begin{cases}\frac{1}{2}(x+2), & x\in[-2,0]\\ 0, & van\end{cases}$.



Slika 4.

b) $E(X)=\int\limits_{-2}^{0}x\left(\frac{1}{2}x+1\right)dx=\int\limits_{-2}^{0}\left(\frac{1}{2}x^2+x\right)dx=\left(\frac{x^3}{6}+\frac{x^2}{2}\right)\Big|_{-2}^{0}=0-\left(\frac{-8}{6}+\frac{4}{2}\right)=\frac{4}{3}-2=-\frac{2}{3}$,

$P(X>-1)=P(-1<X<+\infty)=F(+\infty)-F(-1)=1-\frac{1}{2}(-1+2)=1-\frac{1}{2}=\frac{1}{2}$.

7. Slučajna promenljiva X ima uniformnu raspodelu sa $E(X)=4$ i $D(X)=3$. Naći gustinu za X.

Rešenje:

Ako X ima uniformnu $U(a,b)$ raspodelu, biće $\frac{a+b}{2}=4$ i $\frac{(b-a)^2}{12}=3$, odakle je $a+b=8$, $b-a=6$, tj. $a=1$, $b=7$.

Tražena gustina je $f(x)=\begin{cases}\frac{1}{6}, & x\in[1,7]\\ 0, & izvan\end{cases}$.